

# An introduction to higher energies and sumsets

Shkredov I.D.

Annotation.

*These notes basically contain a material of two mini-courses which were read in Göteborg in April 2015 during the author visit of Chalmers & Göteborg universities and in Beijing in November 2015 during "Chinese-Russian Workshop on Exponential Sums and Sumsets". The article is a short introduction to a new area of Additive Combinatorics which is connected with so-called the higher sumsets as well as with the higher energies. We hope the notes will be helpful for a reader who is interested in the field.*

## 1 Introduction

Let  $\mathbf{G} = (\mathbf{G}, +)$  be a group with the group operation  $+$ . By letters  $A, B, C, \dots$  we will denote arbitrary subsets of the group  $\mathbf{G}$ . Define the sumset and, similarly, the difference set of two sets  $A, B$  as

$$A + B := \{a + b : a \in A, b \in B\} \quad A - B := \{a - b : a \in A, b \in B\}.$$

Of course, one can iterate the sumsets/difference sets, obtaining sums  $A_1 + A_2 + \dots + A_k$  and so on. If  $A_1 = \dots = A_k = A$  then we write  $kA$  for the sumset of  $k$  sets  $A$ . The subject of Additive Combinatorics is any combinatorics which can be expressed with the help of the group operation  $+$ . Typical questions in the field are finding different connections between the sizes of sumsets, different sets, cardinalities of its iterations and so on.

In the notes we investigate higher sumsets and generalized convolutions, which are closely connected with the new object. Also we study higher moments of these convolutions (higher energies), which generalize a classical notion of the additive energy. Such quantities appear in many problems of Additive Combinatorics as well as in Number Theory. In our investigation we use different approaches including basic combinatorics, Fourier analysis and the eigenvalues method to establish basic properties of the higher energies. Also we provide a sequence of applications of the higher energies to Additive Combinatorics and Number Theory.

The first part of the notes (section 3) is based on paper [26] and partially on [35], the second one uses [25] and partially [24], [17]. The last part has roots in paper [27], where nevertheless a "dual" Fourier notation was used. After that the point of view was developed in articles [26], [29], [30] and others. As for applications sections 4, 5 contain modern estimate for the size of the difference sets of convex sets and a new upper bound for Heilbronn's exponential sum. Also we discuss a structural  $E_2, E_3$  result and the best upper bound for the additive energy of a multiplicative subgroup in the last section. Finally, the notes contain some instructive exercises.

## 2 Notation

By  $\mathbf{G} = (\mathbf{G}, +)$  we denote a group with the group operation  $+$ . For a positive integer  $n$ , we set  $[n] = \{1, \dots, n\}$ . All logarithms are base 2. Signs  $\ll$  and  $\gg$  are the usual Vinogradov's symbols. With a slight abuse of notation we use the same letter to denote a set  $S \subseteq \mathbf{G}$  and its characteristic function  $S : \mathbf{G} \rightarrow \{0, 1\}$ , in other words  $S(x) = 1$ ,  $x \in S$  and  $S(x) = 0$  otherwise.

Let  $f, g : \mathbf{G} \rightarrow \mathbb{C}$  be two functions. Put

$$(f * g)(x) := \sum_{y \in \mathbf{G}} f(y)g(x - y) \quad \text{and} \quad (f \circ g)(x) := \sum_{y \in \mathbf{G}} f(y)g(y + x) \quad (1)$$

Clearly,  $(f * g)(x) = (g * f)(x)$  and  $(f \circ g)(x) = (g \circ f)(-x)$ ,  $x \in \mathbf{G}$ . The  $k$ -fold convolution,  $k \in \mathbb{N}$  we denote by  $*_k$ , so  $*_k := (*_{k-1})$ . Put  $\mathbf{E}^+(A, B)$  for the *additive energy* of two sets  $A, B \subseteq \mathbf{G}$  (see e.g. [38]), that is

$$\mathbf{E}^+(A, B) = |\{a_1 + b_1 = a_2 + b_2 : a_1, a_2 \in A, b_1, b_2 \in B\}|.$$

If  $A = B$  then we simply write  $\mathbf{E}^+(A)$  instead of  $\mathbf{E}^+(A, A)$ . Clearly,

$$\mathbf{E}^+(A, B) = \sum_x (A * B)(x)^2 = \sum_x (A \circ B)(x)^2 = \sum_x (A \circ A)(x)(B \circ B)(x).$$

Sumsets and energies are connected by the Cauchy–Schwarz inequality

$$|A|^2|B|^2 \leq \mathbf{E}^+(A, B)|A \pm B|. \quad (2)$$

Note also that

$$\mathbf{E}^+(A, B) \leq \min\{|A|^2|B|, |B|^2|A|, |A|^{3/2}|B|^{3/2}\}. \quad (3)$$

In the same way define the *multiplicative energy* of two sets  $A, B \subseteq \mathbf{G}$

$$\mathbf{E}^\times(A, B) = |\{a_1 b_1 = a_2 b_2 : a_1, a_2 \in A, b_1, b_2 \in B\}|.$$

Certainly, multiplicative energy  $\mathbf{E}^\times(A, B)$  can be expressed in terms of multiplicative convolutions, similar to (1). Usually we will use the additive energy and write  $\mathbf{E}(A, B)$  instead of  $\mathbf{E}^+(A, B)$ . Sometimes we put  $\mathbf{E}(f, g) = \sum_x (f \circ f)(x)(g \circ g)(x)$  for two arbitrary functions  $f, g : \mathbf{G} \rightarrow \mathbb{C}$ .

In the lecture notes we will use Fourier analysis, although it is not main topic of our course. Nevertheless, let us recall the required definitions, for more details see [18].

Let  $\mathbf{G}$  be an abelian group. If  $\mathbf{G}$  is finite then denote by  $N$  the cardinality of  $\mathbf{G}$ . It is well-known [18] that the dual group  $\widehat{\mathbf{G}}$  is isomorphic to  $\mathbf{G}$  in the case. Let  $f$  be a function from  $\mathbf{G}$  to  $\mathbb{C}$ . We denote the Fourier transform of  $f$  by  $\widehat{f}$ ,

$$\widehat{f}(\xi) = \sum_{x \in \mathbf{G}} f(x)e(-\xi \cdot x), \quad (4)$$

where  $e(x) = e^{2\pi i x}$  and  $\xi$  is a homomorphism from  $\widehat{\mathbf{G}}$  to  $\mathbb{R}/\mathbb{Z}$  acting as  $\xi : x \rightarrow \xi \cdot x$ . We rely on the following basic identities

$$\sum_{x \in \mathbf{G}} |f(x)|^2 = \frac{1}{N} \sum_{\xi \in \widehat{\mathbf{G}}} |\widehat{f}(\xi)|^2, \quad (5)$$

$$\sum_{x \in \mathbf{G}} f(x) \overline{g(x)} = \frac{1}{N} \sum_{\xi \in \widehat{\mathbf{G}}} \widehat{f}(\xi) \overline{\widehat{g}(\xi)}, \quad (6)$$

$$\sum_{y \in \mathbf{G}} \left| \sum_{x \in \mathbf{G}} f(x) g(y-x) \right|^2 = \frac{1}{N} \sum_{\xi \in \widehat{\mathbf{G}}} |\widehat{f}(\xi)|^2 |\widehat{g}(\xi)|^2, \quad (7)$$

and

$$f(x) = \frac{1}{N} \sum_{\xi \in \widehat{\mathbf{G}}} \widehat{f}(\xi) e(\xi \cdot x). \quad (8)$$

Further, we have

$$\widehat{f * g} = \widehat{f} \widehat{g} \quad \text{and} \quad \widehat{f \circ g} = \widehat{f}^c \widehat{g} = \overline{\widehat{f}} \widehat{g}, \quad (9)$$

where for a function  $f : \mathbf{G} \rightarrow \mathbb{C}$  we put  $f^c(x) := f(-x)$ . Clearly,  $S$  is the characteristic function of a set iff

$$\widehat{S}(x) = N^{-1} (\overline{\widehat{S}} \circ \widehat{S})(x). \quad (10)$$

In terms of Fourier transform the common additive energy of two sets  $A, B \subseteq \mathbf{G}$ ,  $|\mathbf{G}| < \infty$  can be expressed as follows

$$\mathbb{E}(A, B) = \frac{1}{|\mathbf{G}|} \sum_{\xi \in \widehat{\mathbf{G}}} |\widehat{A}(\xi)|^2 |\widehat{B}(\xi)|^2,$$

see formula (7).

### 3 Higher sumsets

An usual sumset  $A + B$  can be considered as the set of nonempty intersections

$$A + B = \{s \in \mathbf{G} : A \cap (s - B) \neq \emptyset\}.$$

It is more convenient for us to have deal with the symmetric case  $A = B$  and moreover we consider the difference sets instead of the sumsets (basically because the difference sets have more structure than the sumsets). Thus

$$A - A = \{s \in \mathbf{G} : A \cap (A - s) \neq \emptyset\}. \quad (11)$$

If we put  $A_s := A \cap (A - s)$  then the set  $A - A$  is exactly the set of all  $s$  such that  $A_s \neq \emptyset$ . In view of formula (11) a natural multidimensional generalization of the difference sets is

$$\{\vec{s} = (s_1, \dots, s_k) \in \mathbf{G}^k : A \cap (A - s_1) \cap \dots \cap (A - s_k) \neq \emptyset\}. \quad (12)$$

Again, putting  $A_{\vec{s}} := A \cap (A - s_1) \cap \cdots \cap (A - s_k)$ , we see that the set of  $\vec{s}$  which are defined in (11) coincide with the set of vectors  $\vec{s}$  such that  $A_{\vec{s}} \neq \emptyset$ . One can easily check that our multidimensional difference set (or equivalently, the *higher difference set*) is just

$$A^k - \Delta_k(A) = \{(s_1, \dots, s_k) \in \mathbf{G}^k : A \cap (A - s_1) \cap \cdots \cap (A - s_k) \neq \emptyset\},$$

where

$$\Delta(A) = \Delta_k(A) = \{(a, \dots, a) \in \mathbf{G}^k : a \in A\}$$

is a *diagonal set*. We also write  $\Delta(a) = \Delta_k(a) = (a, \dots, a) \in \mathbf{G}^k$  for  $a \in A$ . Thus, any higher difference set is just an usual difference of two but rather specific sets, namely, the Cartesian product  $A^k$  and very thin diagonal set  $\Delta_k(A)$ .

It is well-known that the sumsets/difference sets can be considered as projections of  $A \times A$  along the lines  $y = c - x$  and  $y = c + x$ , correspondingly, onto any of two axes.

**Exercise 1** *Using projections show that for the usual Cantor one-third set  $\mathbf{K}_3$  one has  $\mathbf{K}_3 + \mathbf{K}_3 = [0, 2]$ .*

Similarly, higher difference sets are projections of  $A^k$  along the lines

$$\begin{cases} x_1 = t + c_1 \\ \dots\dots\dots \\ x_k = t + c_k \end{cases}$$

So, the line above intersects  $A^k$  iff  $(c_1, \dots, c_k) \in A^k - \Delta(A)$ . Projections along hyperspaces and connected energies  $\mathbf{T}_k(A)$  which are very popular in Analytical Number Theory will be considered in the next section. What can we say about another projections of intermediate dimensions? The question is widely open.

We continue the section by Ruzsa's triangle inequality, see e.g. [38], which is an important tool of Additive Combinatorics. Interestingly, that our proof (developing some ideas of paper [26]) describes the situation when the triangle inequality is sharp. Namely, the rough equality can be only if  $|B \cap (A - z) - C| \approx |C|$  for many  $z \in A - B$ .

**Lemma 2** *Let  $A, B, C \subseteq \mathbf{G}$  be any sets. Then*

$$|C||A - B| \leq |A \times B - \Delta(C)| \leq |A - C||B - C|. \quad (13)$$

*In particular*

$$|C||A - B| \leq |A - C||B - C|. \quad (14)$$

**Proof.** We give two proofs of (14). The first proof is standard and the second one will follow from (13).

*The first proof.* By the definition of the difference set  $A - B$  for any  $x \in A - B$  there are  $a_x \in A$  and  $b_x \in B$  such that  $x = a_x - b_x$ . Of course it can be exist several pairs  $(a_x, b_x)$  with

the property but we fix just one of them somehow and do not consider another pairs. Now let us define the map

$$\varphi : C \times (A - B) \rightarrow A \times B - \Delta(C) \subseteq (A - C) \times (B - C)$$

by the rule  $\varphi(c, x) = (a_x - c, b_x - c)$ ,  $c \in C$ ,  $x \in A - B$ . It is easy to see that the map is injective. Indeed if we have

$$\varphi(c, x) = (a_x - c, b_x - c) = (a'_x - c', b'_x - c') = \varphi(c', x'), \quad c, c' \in C, x, x' \in A - B \quad (15)$$

then subtracting the second coordinate from the first one, we obtain  $x = a_x - b_x = a'_x - b'_x = x'$ . By our definition of  $a_x, b_x$  for any  $x \in A - B$  there is the only such a pair. Thus  $a_x = a'_x$ ,  $b_x = b'_x$  and we see from (15) that  $c = c'$  as required.

*The second proof.* Now let us prove (13). We have

$$|A \times B - \Delta(C)| = \sum_{q \in A - B} |B \cap (A - q) - C| \geq |A - B||C|.$$

The inequality above is trivial and the identity follows by the projection of points  $(x, y) \in A \times B - \Delta(C)$ ,  $(x, y) = (a - c, b - c)$ ,  $a \in A$ ,  $b \in B$ ,  $c \in C$  onto lines  $q := x - y = a - b \in A - B$ . If  $q$  is fixed we see that the result of the projection is the intersection of the line  $q = x - y$  with our set and moreover the ordinates of the points from the intersection belongs to  $B \cap (A - q) - C$ . It is easy to check that the converse is also true. This concludes the proof.  $\square$

**Exercise 3** Using Ruzsa's triangle inequality, prove Freiman and Pigaev's result  $|A + A|^{3/4} \leq |A - A| \leq |A + A|^{4/3}$ .

The next theorem provides some basic relations between the sizes of the higher dimensional sumsets. The result generalizes Ruzsa's triangle inequality.

**Theorem 4** Let  $k \geq 1$  be a positive integer, and let  $A_1, \dots, A_k, B$  be finite subsets of an abelian group  $\mathbf{G}$ . Further, let  $W, Y \subseteq \mathbf{G}^k$ , and  $X, Z \subseteq \mathbf{G}$ . Then

$$|W \times X||Y - \Delta(Z)| \leq |Y \times W \times Z - \Delta(X)|, \quad (16)$$

$$|A_1 \times \dots \times A_k - \Delta(B)| \leq |A_1 \times \dots \times A_m - \Delta(A_{m+1})||A_{m+1} \times \dots \times A_k - \Delta(B)| \quad (17)$$

for any  $m \in [k]$ . Furthermore, we have

$$|Y \times Z - \Delta(X)| = |Y \times X - \Delta(Z)|. \quad (18)$$

**Proof.** To show the first inequality we apply Ruzsa's argument from the first proof of Lemma 2. For every  $\mathbf{a} \in Y - \Delta(Z)$  choose the smallest element (in any linear order of  $Z$ )  $z \in Z$  such that  $\mathbf{a} = (y_1 - z, \dots, y_k - z)$  for some  $(y_1, \dots, y_k) \in Y$ . Next, observe that the function

$$(\mathbf{a}, \mathbf{w}, x) \mapsto (y_1 - x, \dots, y_k - x, z - x, w_1 - x, \dots, w_k - x),$$

where  $\mathbf{w} = (w_1, \dots, w_k) \in W$  from  $(Y - \Delta(Z)) \times W \times X$  to  $Y \times W \times Z - \Delta(X)$  is injective.

To obtain the second inequality consider the following matrix

$$\mathbf{M} = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 & -1 \\ 0 & 1 & 0 & \dots & 0 & -1 \\ 0 & 0 & 1 & \dots & 0 & -1 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 0 & 1 & -1 \end{pmatrix}$$

Clearly,  $A_1 \times \dots \times A_k - \Delta(B) = \mathbf{Im}(\mathbf{M}|_{A_1 \times \dots \times A_k \times B})$ . Further, non-degenerate transformations of lines does not change the cardinality of the image. Thus, subtracting the  $(m+1)$ th line, we obtain vectors of the form

$$(a_1 - a_{m+1}, \dots, a_m - a_{m+1}, a_{m+1} - b, \dots, a_k - b),$$

which belong to  $(A_1 \times \dots \times A_m - \Delta(A_{m+1})) \times (A_{m+1} \times \dots \times A_k - \Delta(B))$ .

To obtain (18) it is sufficient to show that

$$|Y \times Z - \Delta(X)| \leq |Y \times X - \Delta(Z)|.$$

But the map

$$(y_1 - x, \dots, y_k - x, z - x) \mapsto (y_1 - z, \dots, y_k - z, x - z),$$

where  $(y_1, \dots, y_k) \in Y$ ,  $x \in X$ ,  $z \in Z$  is an injection. This completes the proof.  $\square$

There is another way to prove estimate (17) in spirit of Lemma 2.4 and Corollary 2.5 from [35]. We recall this result, which follows from the definitions.

**Proposition 5** *Let  $k \geq 2$ ,  $m \in [k]$  be positive integers, and let  $A_1, \dots, A_k, B$  be finite subsets of an abelian group. Then*

$$A_1 \times \dots \times A_k - \Delta(B) = \{(x_1, \dots, x_k) : B \cap (A_1 - x_1) \cap \dots \cap (A_k - x_k) \neq \emptyset\} \quad (19)$$

and

$$A_1 \times \dots \times A_k - \Delta(B) = \bigcup_{(x_1, \dots, x_m) \in A_1 \times \dots \times A_m - \Delta(B)} \{(x_1, \dots, x_m)\} \times (A_{m+1} \times \dots \times A_k - \Delta(B \cap (A_1 - x_1) \cap \dots \cap (A_m - x_m))). \quad (20)$$

Indeed, the intersection  $B \cap (A_1 - x_1) \cap \dots \cap (A_k - x_k)$  is nonempty iff, firstly, for  $(x_1, \dots, x_m)$  the intersection  $\mathcal{B} := B \cap (A_1 - x_1) \cap \dots \cap (A_m - x_m)$  is nonempty and, secondly, the intersection of the set  $\mathcal{B}$  with  $(A_{m+1} - x_{m+1}) \cap \dots \cap (A_k - x_k)$  is also nonempty.

**Corollary 6** *We have*

$$\sum_{s \in A - A} |A - A_s| = |A^2 - \Delta(A)|.$$

**Exercise 7** *Prove that*

$$|A^2 + \Delta(A)| = \sum_{s \in A-A} |A + A_s| \geq |A| \cdot \max\{|A + A|, |A - A|\}.$$

Moreover, let  $n, m \geq 1$  be positive integers. Then

$$|A^{n+m} - \Delta(A)| \geq |A|^m |A^n - \Delta(A)|, \quad (21)$$

and

$$|A^{n+m} + \Delta(A)| \geq |A|^m \max\{|A^n + \Delta(A)|, |A^n - \Delta(A)|\}. \quad (22)$$

From (19) one can deduce another characterization of the set  $A^k - \Delta(B)$ , namely,

$$A^k - \Delta(B) = \{X \subseteq \mathbf{G} : |X| = k, B \not\subseteq ((\mathbf{G} \setminus A) - X)\}.$$

Here we used  $X$  to denote a multiset and a corresponding sequence created from  $X$ . Using the characterization it is easy to prove, that if  $A$  is a subset of finite abelian group  $\mathbf{G}$  then there is  $X$ ,  $|X| \sim \frac{N}{|A|} \cdot \log N$  such that  $A + X = \mathbf{G}$ . Indeed, let  $A^c = \mathbf{G} \setminus A$ , and  $k \sim \frac{N}{|A|} \cdot \log N$ . Consider

$$|(A^c)^k - \Delta(A^c)| \leq |A^c|^{k+1} = N^{k+1}(1 - |A|/N)^{k+1} < N^k.$$

Thus, there is a multiset  $X$ ,  $|X| = k$  such that  $A^c \subseteq A - X$ . Whence the set  $-X \cup \{0\}$  has the required property.

The conception of the higher sumsets allows us to introduce a hierarchy of basis of abelian groups, i.e. of sets  $B$  such that  $B \pm B = \mathbf{G}$ . For simplicity, if  $B$  is a basis let us write  $B \oplus_k B$  and  $B \ominus_k B$  for  $B^k + \Delta(B)$  and  $B^k - \Delta(B)$ , respectively.

**Definition 8** *Let  $k \geq 1$  be a positive integer. A subset  $B$  of an abelian group  $\mathbf{G}$  is called basis of depth  $k$  if  $B \ominus_k B = \mathbf{G}^k$ .*

It follows from Theorem 4 that if  $B$  is a basis of depth  $k$  of finite abelian group  $\mathbf{G}$ , then for every set  $A \subseteq \mathbf{G}$

$$|B + A| \geq |A|^{\frac{1}{k+1}} |\mathbf{G}|^{\frac{k}{k+1}}. \quad (23)$$

Taking any one-element  $A$  in formula (23) we obtain, in particular, that  $|B| \geq |\mathbf{G}|^{\frac{k}{k+1}}$  for any basis of depth  $k$ . It is easy to see, using Proposition 5 that every set with  $B$ ,  $|B| > (1 - 1/(k+1))|\mathbf{G}|$  is a basis of depth  $k$  and this inequality is sharp.

If  $S_1, \dots, S_k$  are any sets such that  $S_1 + \dots + S_k = \mathbf{G}$  then the set  $\bigcup_{j=1}^k (\sum_{i \neq j} (S_i - S_i))$  is a basis of depth  $k$  (see Corollary 11 below, the construction can be found in [14]). Let us give another example. Using Weil's bounds for exponential sums we show that quadratic residuals in  $\mathbb{Z}/p\mathbb{Z}$ , for a prime  $p$ , is a basis of depth  $(\frac{1}{2} + o(1)) \log p$ . Clearly, the bound is the best possible up to constants for subsets of  $\mathbb{Z}/p\mathbb{Z}$  of the cardinality less than  $p/2$ .

**Proposition 9** *Let  $p$  be a prime number, and let  $R$  be the set of quadratic residuals. Then  $R$  is the bases of depth  $k$ , where  $k2^k < \sqrt{p}$ .*

**Proof.** Clearly,

$$R(x) = \frac{1}{2} \left( \chi_0(x) + \left( \frac{x}{p} \right) \right),$$

where  $\left( \frac{x}{p} \right)$  is the Legendre symbol and  $\chi_0(x)$  is the main character. Put  $\alpha_0 = 0$ . For all distinct non-zero  $\alpha_1, \dots, \alpha_k$ , we have

$$\begin{aligned} |R \cap (R - \alpha_1) \cap \dots \cap (R - \alpha_k)| &= \frac{1}{2^k} \sum_x \prod_{j=0}^k \left( \chi_0(x) + \left( \frac{x + \alpha_j}{p} \right) \right) \geq \frac{1}{2^k} \left( p - 1 - \sqrt{p} \cdot \sum_{j=2}^k j C_k^j \right) \\ &\geq \frac{1}{2^k} (p - \sqrt{p} \cdot k 2^k) > 0. \end{aligned}$$

We used the well-known Weil bound for exponential sums with multiplicative characters (see e.g. [10]). By formula (19) of Proposition 5 we see that  $R \ominus_k R = \mathbb{Z}_p^k$ .  $\square$

Another consequence of Proposition 9 is that quadratic non-residuals  $Q$  (and, hence, quadratic residuals) have no completion of size smaller than  $(\frac{1}{2} + o(1)) \log p$ , that is a set  $X$  such that  $X + Q = \mathbb{Z}/p\mathbb{Z}$ .

The next proposition is due to N.G. Moshchevitin.

**Proposition 10** *Let  $k_1, k_2$  be positive integers, and  $X_1, \dots, X_{k_1}, Y, Z_1, \dots, Z_{k_2}, W$  be finite subsets of an abelian group. Then we have a bound*

$$\begin{aligned} &|X_1 \times \dots \times X_{k_1} - \Delta(Y)| |Z_1 \times \dots \times Z_{k_2} - \Delta(W)| \leq \\ &\leq |(X_1 - W) \times \dots \times (X_{k_1} - W) \times (Y - Z_1) \times \dots \times (Y - Z_{k_2}) - \Delta(Y - W)|. \end{aligned}$$

**Proof.** It is enough to observe that the map

$$\begin{aligned} &(x_1 - y, \dots, x_{k_1} - y, z_1 - w, \dots, z_{k_2} - w) \mapsto \\ &\mapsto (x_1 - w - (y - w), \dots, x_{k_1} - w - (y - w), y - z_1 - (y - w), \dots, y - z_{k_2} - (y - w)) \end{aligned}$$

where  $x_j \in X_j$ ,  $j \in [k_1]$ ,  $y \in Y$ ,  $z_j \in Z_j$ ,  $j \in [k_2]$ ,  $w \in W$  is injective.  $\square$

In particular, the difference and the sum of two bases of depths  $k_1$  and  $k_2$  is a basis of depth  $k_1 + k_2$ . Let us also formulate a simple identity, which is a consequence of Theorem 4.

**Corollary 11** *Let  $k \geq 2$  be a positive integer, and let  $A_1, \dots, A_k$  be a subsets of a finite abelian group  $\mathbf{G}$ . Then*

$$|A_1 \times \dots \times A_k - \Delta(\mathbf{G})| = |\mathbf{G}| |A_1 \times \dots \times A_{k-1} - \Delta(A_k)|. \quad (24)$$



Thus,  $B$  is a basis of depth  $k$  iff  $B$  is  $(k+1)$ -universal set (see [1]), i.e. a set that is for any  $x_1, \dots, x_{k+1} \in \mathbf{G}$  there is  $z \in \mathbf{G}$  such that  $z + x_1, \dots, z + x_{k+1} \in B$ . A series of very interesting examples of universal sets can be found in [14].

Let  $A, B \subseteq \mathbf{G}$  be two finite sets. The magnification ratio  $R_B[A]$  of the pair  $(A, B)$  (see e.g. [38]) is defined by

$$R_B[A] = \min_{\emptyset \neq Z \subseteq A} \frac{|B + Z|}{|Z|}. \quad (25)$$

We simply write  $R[A]$  for  $R_A[A]$ . Petridis [16] obtained an amazingly short proof of the following fundamental theorem, see book [19].

**Theorem 12** *Let  $A \subseteq \mathbf{G}$  be a finite set, and  $n, m$  be positive integers. Then*

$$|nA - mA| \leq R^{n+m}[A] \cdot |A|.$$

Another beautiful result (which implies Theorem 12) was proven also by Petridis [16].

**Theorem 13** *For any  $A, B, C$ , we have*

$$|B + C + X| \leq R_B[A] \cdot |C + X|,$$

where  $X \subseteq A$  and  $|B + X| = R_B[A]|X|$ .

For a set  $B \subseteq \mathbf{G}^k$  define

$$R_B[A] = \min_{\emptyset \neq Z \subseteq A} \frac{|B + \Delta(Z)|}{|Z|}.$$

In the next two results we assume that  $X \subseteq A$  is such that  $|B + \Delta(X)| = R_B[A]|X|$ . It is easy to see that Petridis argument can be adopted to higher dimensional sumsets, giving a generalization of Theorem 13.

**Theorem 14** *Let  $A \subseteq \mathbf{G}$  and  $B \subseteq \mathbf{G}^k$ . Then for any  $C \subseteq \mathbf{G}$ , we have*

$$|B + \Delta(C + X)| \leq R_B[A] \cdot |C + X|.$$

A consequence of Theorem 14, we obtain a generalization of the sum version of the triangle inequality (see, e.g. [19]).

**Corollary 15** *Let  $k$  be a positive integer,  $A, C \subseteq \mathbf{G}$  and  $B \subseteq \mathbf{G}^k$  be finite sets. Then*

$$|A||B + \Delta(C)| \leq |B + \Delta(A)||A + C|.$$

**Proof.** Using Theorem 14, we have

$$|B + \Delta(C)| \leq |B + \Delta(C + X)| \leq R_B[A] \cdot |C + X| \leq \frac{|B + \Delta(A)|}{|A|} |A + C|$$

and the result follows.  $\square$

Thus, we have the following sum-bases analog of inequality (23).

**Corollary 16** *Let  $k$  be a positive integer, and  $B \oplus_k B = \mathbf{G}^k$ . Then for any set  $A \subseteq \mathbf{G}$ , we have*

$$|B + A| \geq |A|^{\frac{1}{k+1}} |\mathbf{G}|^{\frac{k}{k+1}}.$$

It is well known that Croot–Sisask [3] Lemma on almost periodicity of convolutions has become a central tool of Additive Combinatorics, see applications, say, in [3], [23], [21], [22]. We conclude the section, showing that some large subsets of  $A_{\vec{s}} - a_{\vec{s}}$ ,  $a \in A_{\vec{s}}$  for typical  $\vec{s}$  are the sets of almost periods which appear in the arguments of Croot and Sisask. It demonstrates the importance of the higher sumsets once more time. For simplicity we consider just a symmetric case.

**Theorem 17** *Let  $\epsilon \in (0, 1)$ ,  $K \geq 1$  be real numbers and  $p$  be a positive integer. Let also  $A \subseteq \mathbf{G}$  be sets with  $|A - A| \leq K|A|$  and let  $f \in L_p(\mathbf{G})$  be an arbitrary function. Then there is a  $a \in A$  and a set  $T \subseteq A$ ,  $|T| \geq |A|(2K)^{-O(\epsilon^{-2}p)}$  such that for all  $t \in T$  – a one has*

$$\|(f * A)(x + t) - (f * A)(x)\|_{L_p(\mathbf{G}, x)} \leq \epsilon \|f\|_{L_p(\mathbf{G})} \cdot |A|^{1/p}. \quad (26)$$

**Proof.** Write  $\mu_A(x) = A(x)/|A|$ . Let  $k$  be a natural parameter,  $k = O(\epsilon^{-2}p)$ . Take  $k$  points  $x_1, \dots, x_k \in A$  uniformly and random and put  $X_j(y) = f(y + x_j) - (f * \mu_A)(y)$ . For any fixed  $y$  the random variables  $X_j(y)$  are independent with zero mean and variance at most  $(|f|^2 * \mu_A)(y)$ . By the Khinchin inequality for sums of independent random variables, we get

$$\left\| \sum_{j=1}^k X_j(y) \right\|_{L_p(\mu_A^k)} \ll (pk(|f|^2 * \mu_A)(y))^{1/2}.$$

Taking  $p$ th power, dividing by  $k^p$ , integrating over  $y$  and using Hölder inequality for  $L_p(y - A)$ , which gives us  $(|f|^2 * \mu_A)^{p/2}(y) \leq (|f|^p * \mu_A)(y)$ , we obtain

$$\int \int \left| \frac{1}{k} \sum_{j=1}^k f(y + x_j) - (f * \mu_A)(y) \right|^p dy d\mu_A^k(\vec{x}) \ll (pk^{-1} \|f\|_{L_p(\mathbf{G})}^2)^{p/2}.$$

Here  $\vec{x} = (x_1, \dots, x_k)$ . Applying the Hölder inequality again and recalling that  $k = O(\epsilon^{-2}p)$ , we have

$$\int \left\| \frac{1}{k} \sum_{j=1}^k f(y + x_j) - (f * \mu_A)(y) \right\|_{L_p(\mathbf{G}, y)} d\mu_A^k(\vec{x}) \leq \epsilon \|f\|_{L_p(\mathbf{G})}/4. \quad (27)$$

From estimate (27) it follows that the set  $L$  of all  $\vec{x} \in A^k$  such that the norm in the inequality less than  $\epsilon \|f\|_{L_p(\mathbf{G})}/2$  has measure  $\mu_A^k(L) \geq 1/2$ . Putting  $\mu_{\vec{x}}$  equals the probability measure sitting on the points  $x_1, \dots, x_k$ , we obtain  $(\mu_{\vec{x}} * f)(y) = \frac{1}{k} \sum_{j=1}^k f(y + x_j)$  and then (27) says us

$$\|\mu_{\vec{x}}(y) - (f * \mu_A)(y)\|_{L_p(\mathbf{G}, y)} \leq \epsilon \|f\|_{L_p(\mathbf{G})}/2 \quad (28)$$

provided by  $\vec{x} \in L$ .

Now we can construct our set of almost periods. Clearly,

$$A_{\vec{s}} = \{a \in A : \Delta(a) + \vec{s} \in A^k\}, \quad \vec{s} \in A^k - \Delta(A).$$

Thus, put

$$A'_{\vec{s}} = \{a \in A : \Delta(a) + \vec{s} \in L\} \subseteq A_{\vec{s}}.$$

We claim that any set  $A'_{\vec{s}}$ ,  $\vec{s} \in L$  is a set of almost periods (it corresponds to the arguments of T. Sanders from [21], say). Indeed, by the definition of the set  $L$ , see formula (28), we have for any  $a \in A'_{\vec{s}}$  that

$$\|\mu_{\Delta(a)+\vec{s}}(y) - (f * \mu_A)(y)\|_{L_p(\mathbf{G}, y)} = \|\mu_{\vec{s}}(y + a) - (f * \mu_A)(y)\|_{L_p(\mathbf{G}, y)} \leq \epsilon \|f\|_{L_p(\mathbf{G})}/2 \quad (29)$$

and

$$\|\mu_{\vec{s}}(y) - (f * \mu_A)(y)\|_{L_p(\mathbf{G}, y)} \leq \epsilon \|f\|_{L_p(\mathbf{G})}/2. \quad (30)$$

Thus by the triangle inequality and because of any shift preserves  $L_p$ -norm, one has

$$\|(f * \mu_A)(y - a) - (f * \mu_A)(y)\|_{L_p(\mathbf{G}, y)} = \|(f * \mu_A)(y + a) - (f * \mu_A)(y)\|_{L_p(\mathbf{G}, y)} \leq \epsilon \|f\|_{L_p(\mathbf{G})}.$$

Finally, we show that there is large set  $A'_{\vec{s}}$  with  $\vec{s} \in L$ . Indeed, clearly,  $A'_{\vec{s}}(a) = A(a)L(\vec{s} + \Delta(a))$  and hence

$$\sum_{\vec{s} \in L} |A'_{\vec{s}}| = \sum_a A(a)(L \circ L)(\Delta(a)) = \sum_{\vec{z}} \Delta(A)(\vec{z})(L \circ L)(\vec{z}) := \sigma.$$

By (2), we have

$$\frac{|L|^2|A|^2}{|A^k - \Delta(A)|} \leq \frac{|L|^2|A|^2}{|L - \Delta(A)|} \leq \mathbb{E}(\Delta(A), L) = \sum_{\vec{z}} (\Delta(A) \circ \Delta(A))(\vec{z})(L \circ L)(\vec{z}) \leq |A|\sigma,$$

and thus there is  $\vec{s} \in L$  such that

$$|A'_{\vec{s}}| \geq |A| \cdot \frac{|L|}{|A^k - \Delta(A)|} \geq 2^{-1}|A| \cdot \frac{|A|^k}{|A^k - \Delta(A)|} \geq 2^{-1}K^{-k}|A| \quad (31)$$

as required.

The arguments above, actually, demonstrate that one can drop the requirement  $\vec{s} \in L$  (we thanks T. Schoen who show us the proof). Indeed, just take any  $a \in A'_s$  for  $\vec{s} \in A^k - \Delta(A)$  and nonempty  $A'_s$  and after that check, using calculations in (29), (30) that the set  $T := A'_s - a$  is a set of almost periods. Moreover, the choice of  $T$  allows us to find large  $A'_s$  even simpler. Indeed,  $\sum_{\vec{s}} |A'_s| = |L||A| \geq |A|^{k+1}/2$  and thus lower bound (31) holds immediately. This completes the proof.  $\square$

## 4 Higher energies

In the section we develop the functional point of view on the higher sumsets. First of all let us define the *generalized convolutions*. Let  $k$  be a positive integer and  $f_1, \dots, f_{k+1} : \mathbf{G} \rightarrow \mathbb{C}$  be any functions. Denote by

$$\mathcal{C}_{k+1}(f_1, \dots, f_{k+1})(x_1, \dots, x_k)$$

the function

$$\mathcal{C}_{k+1}(f_1, \dots, f_{k+1})(x_1, \dots, x_k) = \sum_z f_1(z) f_2(z + x_1) \dots f_{k+1}(z + x_k).$$

Thus,  $\mathcal{C}_2(f_1, f_2)(x) = (f_1 \circ f_2)(x)$ . If  $f_1 = \dots = f_{k+1} = f$  then write  $\mathcal{C}_{k+1}(f)(x_1, \dots, x_k)$  for  $\mathcal{C}_{k+1}(f_1, \dots, f_{k+1})(x_1, \dots, x_k)$ . It is easy to see that

$$\text{supp } \mathcal{C}_{k+1}(A) = A^k - \Delta(A),$$

that is a higher sumset from the previous section. Hence higher sumsets appear naturally as supports of these generalized convolutions.

Let us make a general remark about the functions  $\mathcal{C}_{k+1}(f_1, \dots, f_{k+1})(x_1, \dots, x_k)$ . Suppose that  $l, k \geq 2$  be positive integers and  $\mathbf{F} = (f_{ij})$ ,  $i = 0, \dots, l-1$ ;  $j = 0, \dots, k-1$  be a functional matrix,  $f_{ij} : \mathbf{G} \rightarrow \mathbb{C}$ . Let  $R_0, \dots, R_{l-1}$  and  $C_0, \dots, C_{k-1}$  be rows and columns of the matrix, correspondingly. The following commutative relation holds.

**Lemma 18** *For any positive integers  $l, k \geq 2$ , we have*

$$\mathcal{C}_l(\mathcal{C}_k(R_0), \dots, \mathcal{C}_k(R_{l-1})) = \mathcal{C}_k(\mathcal{C}_l(C_0), \dots, \mathcal{C}_l(C_{k-1})). \quad (32)$$

**Proof.** Let  $y^{(i)} = (y_{i1}, \dots, y_{i(k-1)})$ ,  $i \in [l-1]$ , and  $y_{(j)} = (y_{1j}, \dots, y_{(l-1)j})$ ,  $j \in [k-1]$ . Put also  $y_{0j} = 0$ ,  $j = 0, \dots, k-1$ ,  $y_{i0} = 0$ ,  $i = 1, \dots, l-1$  and  $x_0 = 0$ . We have

$$\begin{aligned} & \mathcal{C}_l(\mathcal{C}_k(R_0), \dots, \mathcal{C}_k(R_{l-1}))(y^{(1)}, \dots, y^{(l-1)}) = \\ &= \sum_{x_1, \dots, x_{k-1}} \mathcal{C}_k(R_0)(x_1, \dots, x_{k-1}) \mathcal{C}_k(R_1)(x_1 + y_{11}, \dots, x_{k-1} + y_{1(k-1)}) \dots \\ & \dots \mathcal{C}_k(R_{l-1})(x_1 + y_{(l-1)1}, \dots, x_{k-1} + y_{(l-1)(k-1)}) = \sum_{x_0, \dots, x_{k-1}} \sum_{z_0, \dots, z_{l-1}} \prod_{i=0}^{l-1} \prod_{j=0}^{k-1} f_{ij}(x_j + y_{ij} + z_i). \end{aligned}$$

Changing the summation, we obtain

$$\begin{aligned}
& \mathcal{C}_l(\mathcal{C}_k(R_0), \dots, \mathcal{C}_k(R_{l-1}))(y^{(1)}, \dots, y^{(l-1)}) = \\
& = \sum_{z_1, \dots, z_{l-1}} \mathcal{C}_l(C_0)(z_1, \dots, z_{l-1}) \mathcal{C}_l(C_1)(z_1 + y_{11}, \dots, z_{l-1} + y_{(l-1)1}) \dots \\
& \dots \mathcal{C}_l(C_{l-1})(z_1 + y_{1(k-1)}, \dots, z_{l-1} + y_{(l-1)(k-1)}) = \mathcal{C}_k(\mathcal{C}_l(C_0), \dots, \mathcal{C}_l(C_{k-1}))(y_{(1)}, \dots, y_{(k-1)}).
\end{aligned}$$

as required.  $\square$

**Corollary 19** *For any functions the following holds*

$$\begin{aligned}
& \sum_{x_1, \dots, x_{l-1}} \mathcal{C}_l(f_0, \dots, f_{l-1})(x_1, \dots, x_{l-1}) \mathcal{C}_l(g_0, \dots, g_{l-1})(x_1, \dots, x_{l-1}) = \\
& = \sum_z (f_0 \circ g_0)(z) \dots (f_{l-1} \circ g_{l-1})(z) \quad \textbf{(scalar product)}, \tag{33}
\end{aligned}$$

moreover

$$\begin{aligned}
& \sum_{x_1, \dots, x_{l-1}} \mathcal{C}_l(f_0)(x_1, \dots, x_{l-1}) \dots \mathcal{C}_l(f_{k-1})(x_1, \dots, x_{l-1}) = \\
& = \sum_{y_1, \dots, y_{k-1}} \mathcal{C}_k^l(f_0, \dots, f_{k-1})(y_1, \dots, y_{k-1}) \quad \textbf{(multi-scalar product)}, \tag{34}
\end{aligned}$$

and

$$\begin{aligned}
& \sum_{x_1, \dots, x_{l-1}} \mathcal{C}_l(f_0)(x_1, \dots, x_{l-1}) (\mathcal{C}_l(f_1) \circ \dots \circ \mathcal{C}_l(f_{k-1}))(x_1, \dots, x_{l-1}) = \\
& = \sum_z (f_0 \circ \dots \circ f_{k-1})^l(z) \quad (\sigma_k \textbf{ for } \mathcal{C}_l). \tag{35}
\end{aligned}$$

**Proof.** Take  $k = 2$  in (32). Thus  $\mathbf{F}$  is a  $l \times 2$  matrix in the case. We have

$$\mathcal{C}_l(f_0 \circ g_0, \dots, f_{l-1} \circ g_{l-1})(x_1, \dots, x_{l-1}) = (\mathcal{C}_l(f_0, \dots, f_{l-1}) \circ \mathcal{C}_l(g_0, \dots, g_{l-1}))(x_1, \dots, x_{l-1}).$$

Putting  $x_j = 0$ ,  $j \in [l-1]$ , we obtain (33). Applying the last formula  $(k-2)$  times and after that formula (33), we get (35). Finally, taking  $\mathbf{F}_{ij} = f_j$ ,  $i = 0, \dots, l-1$ ;  $j = 0, \dots, k-1$  and putting all variables in (32) equal zero, we obtain (34). This completes the proof.  $\square$

Now we can make the main definition of the section.

**Definition 20** *Let  $k, l$  be any positive integers, and  $A \subseteq \mathbf{G}$  be a set. Then*

$$\mathbf{E}_{k,l}(A) := \sum_{x_1, \dots, x_{k-1}} \mathcal{C}_k^l(A)(x_1, \dots, x_{k-1}).$$

*If  $k = 2$  then we write  $\mathbf{E}_l(A)$  for  $\mathbf{E}_{2,l}(A)$ . Clearly,*

$$\mathbf{E}_{k+1}(A) = \mathbf{E}(\Delta_k(A), A^k).$$

If  $l = 1$  then, obviously,  $E_{k,1}(A) = |A|^{k+1}$  and if  $k = 1$  then put for symmetry  $E_{1,l}(A) = |A|^{l+1}$ . Thus, the cardinality of a set can be considered as a degenerate energy. Sometimes we need in energies  $E_\alpha(A)$  for real  $\alpha$ , that is  $E_\alpha(A) = \sum_x (A \circ A)^\alpha(x)$ .

**Corollary 21** *For any positive integers  $k, l$  one has  $E_{k,l}(A) = E_{l,k}(A)$ . In particular*

$$\sum_{(\alpha, \beta) \in A^2 - \Delta(A)} \mathcal{C}_3^2(\alpha, \beta) = E_{3,2}(A) = E_{2,3}(A) = E_3(A). \quad (36)$$

Now let us prove a lemma on the connection of the higher energies of the sets  $A_{\vec{s}}$  and  $A$ .

**Lemma 22** *Let  $A \subseteq \mathbf{G}$  be a set,  $l \geq 1$ ,  $k \geq 2$  be positive integers. Then*

$$\sum_{\vec{s}_1, \dots, \vec{s}_k} \sum_{z_1, \dots, z_{k-1}} \mathcal{C}_k^l(A_{\vec{s}_1}, \dots, A_{\vec{s}_k})(z_1, \dots, z_{k-1}) = \sum_{x_1, \dots, x_{l-1}} \mathcal{C}_l^{\|\vec{s}\|+k}(A)(x_1, \dots, x_{l-1}), \quad (37)$$

where  $\|\vec{s}\| = \sum_{j=1}^k |\vec{s}_j|$ . In particular,

$$\sum_{\vec{s}_1, \dots, \vec{s}_k} \sum_{z_1, \dots, z_{k-1}} \mathcal{C}_k(A_{\vec{s}_1}, \dots, A_{\vec{s}_k})(z_1, \dots, z_{k-1}) = |A|^{\|\vec{s}\|+k}, \quad (38)$$

and

$$\sum_{\vec{s}_1, \dots, \vec{s}_k} \sum_{z_1, \dots, z_{k-1}} \mathcal{C}_k^2(A_{\vec{s}_1}, \dots, A_{\vec{s}_k})(z_1, \dots, z_{k-1}) = \sum_{\vec{s}_1, \dots, \vec{s}_k} E_k(A_{\vec{s}_1}, \dots, A_{\vec{s}_k}) = E_{\|\vec{s}\|+k}(A). \quad (39)$$

**Proof.** Let us put  $z_0 = 0$  for convenience. We have

$$\begin{aligned} \sum_{\vec{s}_1, \dots, \vec{s}_k} \sum_{z_1, \dots, z_{k-1}} \mathcal{C}_k^l(A_{\vec{s}_1}, \dots, A_{\vec{s}_k})(z_1, \dots, z_{k-1}) &= \sum_{\vec{s}_1, \dots, \vec{s}_k} \sum_{z_1, \dots, z_{k-1}} \sum_{w_1, \dots, w_l} \prod_{j=1}^l \prod_{i=1}^k A_{\vec{s}_i}(w_j + z_{i-1}) \\ &= \sum_{w_1, \dots, w_l} \sum_{z_1, \dots, z_{k-1}} \mathcal{C}_k^{\|\vec{s}\|}(A)(w_2 - w_1, \dots, w_l - w_1) \prod_{j=1}^l \prod_{i=1}^k A(w_j + z_{i-1}) = \\ &= \sum_{w_1, \dots, w_l} \mathcal{C}_k^{\|\vec{s}\|+k-1}(A)(w_2 - w_1, \dots, w_l - w_1) A(w_1) \dots A(w_l) = \sum_{x_1, \dots, x_{l-1}} \mathcal{C}_l^{\|\vec{s}\|+k}(A)(x_1, \dots, x_{l-1}), \end{aligned} \quad (40)$$

because each component of any vector  $\vec{s}_i$  appears at formula (40) exactly  $l$  times. This completes the proof.  $\square$

**Corollary 23** *For any  $A \subseteq \mathbf{G}$ , we have*

$$\sum_s E(A, A_s) = E_3(A), \quad \sum_{s,t} E(A_s, A_t) = E_4(A). \quad (41)$$

Now let us look at the higher energies from the point of view of Fourier analysis. First of all, let us define a classical generalization of the additive energy of a set, see e.g. [13]. Let

$$\mathsf{T}_k(A) := |\{a_1 + \dots + a_k = a'_1 + \dots + a'_k : a_1, \dots, a_k, a'_1, \dots, a'_k \in A\}|.$$

Using formula (7) it can be shown that

$$\mathsf{T}_k(A) = \frac{1}{N} \sum_{\xi} |\widehat{A}(\xi)|^{2k}.$$

Let also

$$\sigma_k(A) := (A *_k A)(0) = |\{a_1 + \dots + a_k = 0 : a_1, \dots, a_k \in A\}|.$$

Quantities  $\mathsf{E}_k(A)$  and  $\mathsf{T}_k(A)$  are "dual" in some sense. For example in [30], Note 6.6 (see also [26]) it was proved that

$$\left( \frac{\mathsf{E}_{3/2}(A)}{|A|} \right)^{2k} \leq \mathsf{E}_k(A) \mathsf{T}_k(A), \quad (42)$$

provided by  $k$  is even. Moreover, from (4)—(8), (10) it follows that

$$\tilde{\mathsf{E}}_{2k}(\widehat{A}) := \sum_x (\widetilde{\widehat{A}} \circ \widehat{A})^k(x) (\widehat{A} \circ \widetilde{\widehat{A}})^k(x) = N^{2k+1} \mathsf{T}_k(A), \quad (43)$$

and

$$\mathsf{T}_k(|\widehat{A}|^2) = N^{2k-1} \mathsf{E}_{2k}(A). \quad (44)$$

Another dual formulae can be find in [26]. We give just an example.

**Exercise 24** *Let  $A$  be a subset of an abelian group. Then for every  $k \in \mathbb{N}$ , we have*

$$|A|^{2k} \leq \mathsf{E}_k(A) \cdot \sigma_k(A - A), \quad |A|^{4k} \leq \mathsf{E}_{2k}(A) \cdot \mathsf{T}_k(A + A), \quad (45)$$

and

$$|A|^{2k+4} \leq \mathsf{E}_{k+2}(A) \cdot \mathsf{E}_k(A - A), \quad |A|^{2k+4} \leq \mathsf{E}_{k+2}(A) \cdot \mathsf{E}_k(A + A). \quad (46)$$

Now we are ready to obtain a first application of the method of higher energies.

Let  $A = \{a_1, \dots, a_n\}$ ,  $a_i < a_{i+1}$  be a set of real numbers. We say that  $A$  is *convex* if

$$a_{i+1} - a_i > a_i - a_{i-1}$$

for every  $i = 2, \dots, n-1$ . Hegyvári [8], answering a question of Erdős, proved that if  $A$  is convex then

$$|A + A| \gg |A| \log |A| / \log \log |A|.$$

This result was later improved by many authors. Konyagin [12] and Garaev [5] showed independently that the additive energy  $E(A) = E(A, A)$  of a convex set is  $\ll |A|^{5/2}$ , which immediately implies that

$$|A \pm A| \gg |A|^{3/2}.$$

Elekes, Nathanson and Ruzsa [4] proved that if  $A$  is convex then

$$|A + B| \gg |A|^{3/2}$$

for every set  $B$  with  $|B| = |A|$ . Finally, Solymosi [36] generalized the above inequality, showing that if  $A$  is a set with distinct consecutive differences i.e.  $a_{i+1} - a_i = a_{j+1} - a_j$  implies  $i = j$  then

$$|A + B| \gg |A||B|^{1/2} \quad (47)$$

for every set  $B$ .

In [25] the following theorem was proved.

**Theorem 25** *Let  $A$  be a convex set. Then*

$$|A - A| \gg |A|^{8/5} \log^{-2/5} |A|, \quad (48)$$

and

$$|A + A| \gg |A|^{14/9} \log^{-2/3} |A|, \quad |A + A|^3 |A - A|^2 \log^2 |A| \gg |A|^8. \quad (49)$$

For simplicity we have dealt just with the difference case, so we will prove estimate (48). The best result for the *sumsets* of convex sets can be found in [33].

We need in several lemmas. The first one is the Szemerédi–Trotter theorem [37], see also [38]. We call a set  $\mathcal{L}$  of continuous plane curves a *pseudo-line system* if any two members of  $\mathcal{L}$  share at most one point in common. Define the *number of indices*  $\mathcal{I}(\mathcal{P}, \mathcal{L})$  as  $\mathcal{I}(\mathcal{P}, \mathcal{L}) = |\{(p, l) \in \mathcal{P} \times \mathcal{L} : p \in l\}|$ .

**Theorem 26** *Let  $\mathcal{P}$  be a set of points and let  $\mathcal{L}$  be a pseudo-line system. Then*

$$\mathcal{I}(\mathcal{P}, \mathcal{L}) \ll |\mathcal{P}|^{2/3} |\mathcal{L}|^{2/3} + |\mathcal{P}| + |\mathcal{L}|.$$

The next definition is basically from [4].

**Definition 27** *Let  $A \subset \mathbb{R}$  be a finite set. Put*

$$d(A) = \inf_f \min_{C \neq \emptyset} \frac{|f(A) + C|^2}{|A||C|}, \quad (50)$$

where the infimum over  $f$  is taken over all convex/concave functions.

It is easy to see that  $1 \leq d(A) \leq |A|$ .



**Exercise 28** For simplicity fix  $f$  in (50) and show that the minimum is attained.

We need in the following lemma, see [17], previous results of similar form were proved in [15], [26].

**Lemma 29** Let  $A, B \subset \mathbb{R}$  be finite sets. Then for all  $\tau > 0$  one has

$$|\{x : (A \circ B)(x) \geq \tau\}| \ll d(A) \cdot \frac{|A||B|^2}{\tau^3}. \quad (51)$$

**Proof.** Without loosing of generality we can suppose that  $\tau$  is a positive integer. Let  $C$  be a set, where the minimum in (50) is attained. Take  $x \in \mathbb{R}$  such that  $(A \circ B)(x) \geq \tau$ . Then  $x = a_1 - b_1 = \dots = a_\tau - b_\tau$  for some  $a_j \in A, b_j \in B$ . Hence for any  $c \in C$ , we have

$$c = -f(x + b_j) + f(a_j) + c, \quad j \in [\tau]. \quad (52)$$

Consider the family of convex curves  $\mathcal{L} = \{l_{s,b}\}$ , where  $s \in f(A) + C, b \in B$ , defining by the equation

$$l_{s,b} = \{(x, y) : y = -f(x + b) + s\}.$$

Clearly,  $|\mathcal{L}| = |f(A) + C||B|$ . Let also  $\mathcal{P}$  be the set of all intersecting points, defining by the curves. In the terms identity (52) says us that the point  $(x, c)$  belongs to the  $\tau$  curves  $l_{f(a_j)+c, b_j}$ ,  $j \in [\tau]$ . Hence  $(x, c) \in \mathcal{P}_t$ . The set of curves  $\mathcal{L}$  satisfies the conditions of the Szemerédi–Trotter Theorem. Indeed, if  $(\alpha, \beta) \in l_{s,b} \cap l_{s',b'}$  then  $h(x) := f(x + b') - f(x + b) + s - s' = 0$ . By the assumption  $f(x)$  is a convex/concave function. Hence  $h(x)$  is a monotone function. It follows that the equation  $h(x) = 0$  has at most one solution. Thus by Theorem 26, we have

$$\begin{aligned} |C| \cdot |\{x : (A \circ B)(x) \geq \tau\}| &\leq |\mathcal{P}_t| \ll \frac{|f(A) + C|^2 |B|^2}{\tau^3} + \frac{|f(A) + C||B|}{\tau} \ll \\ &\ll \frac{|f(A) + C|^2 |B|^2}{\tau^3}. \end{aligned}$$

In the last formula, we have used a trivial inequality

$$\tau^2 \leq (\min\{|A|, |B|\})^2 \leq |A||B| \leq |f(A) + C||B|.$$

This completes the proof.  $\square$

**Exercise 30** Let  $A$  be a convex set and  $A' \subseteq A$  be its subset such that  $|A'| \gg |A|$ . Then show that  $d(A') \ll 1$ . In particular it implies that for an arbitrary  $B$  the following holds

$$|A' + B| \gg |A||B|^{1/2}. \quad (53)$$

Let  $A$  be a convex set and  $B$  be an arbitrary set. Order the elements  $s \in B - A$  such that  $(A \circ B)(s_1) \geq (A \circ B)(s_2) \geq \dots \geq (A \circ B)(s_t)$ ,  $t = |B - A|$ . The next lemma was proved in [5], say, and is immediate consequence of Lemma 29.

**Lemma 31** *Let  $A$  be a convex set and  $B$  be an arbitrary set. Then for every  $j \geq 1$  we have*

$$(A \circ B)(s_j) \ll (|A||B|^2)^{1/3} j^{-1/3}.$$

**Corollary 32** *Let  $A$  be a convex set. Then*

$$E_3(A) \ll |A|^3 \log |A|,$$

and

$$E(A, B) \ll |A||B|^{3/2}.$$

**Proof.** To obtain the first formula, just use Lemma 31 with  $B = A$

$$E_3(A) = \sum_x (A \circ A)^3(x) = \sum_{j \geq 1} (A \circ A)^3(s_j) \ll |A|^3 \sum_{j=1}^{|A|} j^{-1} \ll |A|^3 \log |A|.$$

To get the second estimate, choose a parameter  $\tau = |B|^{1/2}$  and use Lemma 31 again

$$\begin{aligned} E(A, B) &= \sum_x (A \circ B)^2(x) = \sum_{x : (A \circ B)(x) < \tau} (A \circ B)^2(x) + \sum_{x : (A \circ B)(x) \geq \tau} (A \circ B)^2(x) \leq \\ &\leq \tau |A||B| + \sum_{x : (A \circ B)(x) \geq \tau} (A \circ B)^2(x) \leq \tau |A||B| + \sum_{j \geq 1 : (A \circ B)(s_j) \geq \tau} (A \circ B)^2(s_j) \ll \\ &\ll \tau |A||B| + (|A||B|^2)^{2/3} \sum_{j=1}^{|A||B|^2 \tau^{-3}} j^{-2/3} \ll \tau |A||B| + \frac{|A||B|^2}{\tau} \ll |A||B|^{3/2}. \end{aligned}$$

This completes the proof.  $\square$

Using the higher energies we obtain a result from [25].

**Theorem 33** *Let  $A$  be a convex set. Then*

$$|A - A| \gg |A|^{8/5} \log^{-2/5} |A|. \quad (54)$$

**Proof.** Put  $D = A - A$ . We have

$$|A|^2 = \sum_{s \in D} |A_s| \leq 2 \sum_{s \in P} |A_s|, \quad (55)$$

where  $P := \{s \in D : |A_s| \geq |A|^2/(2|D|)\}$ . Thus, using (55) and the Cauchy-Schwarz inequality, we obtain

$$2^{-1}|A|^3 \leq |A| \sum_{s \in P} |A_s| = \sum_{s \in P} \sum_x (A \circ A_s)(x) \leq \sum_{s \in P} E^{1/2}(A, A_s) |A - A_s|^{1/2}.$$

Applying the Cauchy–Schwarz inequality once more time, we get in view of Corollary 23 that

$$|A|^6 \ll \sum_s E(A, A_s) \cdot \sum_{s \in P} |A - A_s| = E_3(A) \cdot \sum_{s \in P} |A - A_s|.$$

By Katz–Koester trick [11], that is inclusion  $A - A_s \subseteq D \cap (D + s)$ , we have

$$|A|^6 \ll E_3(A) \cdot \sum_{s \in P} |D_s|.$$

But by the definition of the set  $P$ , we know that  $|A_s| \geq |A|^2/(2|D|)$  for all  $s \in P$ . Hence

$$|A|^8 \ll |D| E_3(A) \cdot \sum_s |A_s| |D_s| = |D| E_3(A) E(A, D). \quad (56)$$

Finally, applying Corollary 32, we get

$$|A|^8 \ll |D| |A|^3 \log |A| \cdot |A| |D|^{3/2}.$$

Using some algebra, we obtain the result. This completes the proof.  $\square$

In the next section we will see that inequality (56) is just a consequence of a simple fact about some specific operators.

## 5 Higher energies and eigenvalues of some operators

Now we introduce some operators, which firstly appeared in [27] in a dual form and were connected with some restrictions problems of Fourier analysis. Nevertheless, our definition below does not use any Fourier analysis and is a purely combinatorial.

Let  $g : \mathbf{G} \rightarrow \mathbb{C}$  be a function, and  $A \subseteq \mathbf{G}$  be a finite set. By  $T_A^g$  denote the  $(|A| \times |A|)$  matrix with indices in the set  $A$

$$T_A^g(x, y) = g(x - y) A(x) A(y), \quad (57)$$

and, similarly, put

$$\tilde{T}_A^g(x, y) = g(x + y) A(x) A(y). \quad (58)$$

It is easy to see that  $T_A^g$  is hermitian iff  $\overline{g(-x)} = g(x)$  and  $\tilde{T}_A^g$  is hermitian iff  $g$  is a real function. Basically, we shall deal with  $T_A^g$ . We have

$$(T_A^g f)(x) = A(x) (g * f)(x). \quad (59)$$

In particular, the corresponding action of  $T_A^g$  is

$$\langle T_A^g a, b \rangle = \sum_z g(z) (\bar{b} \circ a)(z). \quad (60)$$

for any functions  $a, b : A \rightarrow \mathbb{C}$ . In the case  $\overline{g(-x)} = g(x)$  by  $\text{Spec}(\mathbf{T}_A^g)$  we denote the spectrum of the hermitian operator  $\mathbf{T}_A^g$  (which is automatically real for hermitian matrices)

$$\text{Spec}(\mathbf{T}_A^g) = \{\mu_1 \geq \mu_2 \geq \dots \geq \mu_{|A|}\}. \quad (61)$$

Write  $\{f\}_\alpha$ ,  $\alpha \in [|A|]$  for the corresponding orthonormal eigenfunctions. We call  $\mu_1$  as the main eigenvalue and  $f_1$  as the main function. By the spectral theorem for hermitian matrices, see e.g. [9], we have

$$\mathbf{T}_A^{g_1}(x, y) = \sum_{\alpha=1}^{|A|} \mu_\alpha f_\alpha(x) \overline{f_\alpha(y)}. \quad (62)$$

Counting the trace of the operator  $\mathbf{T}_A^g$  and the trace of  $\mathbf{T}_A^g(\mathbf{T}_A^g)^*$  one can easily obtains the formulae

$$\text{tr}(\mathbf{T}_A^g) = \sum_{\alpha=1}^{|A|} \mu_\alpha = g(0)|A|. \quad (63)$$

$$\text{tr}(\mathbf{T}_A^g(\mathbf{T}_A^g)^*) = \sum_{\alpha=1}^{|A|} |\mu_\alpha|^2 = \sum_{x,y} |\mathbf{T}_A^g(x, y)|^2 = \sum_z |g(z)|^2 (A \circ A)(z). \quad (64)$$

Using a particular case of the variational principle [9] one can estimate the the main eigenvalue of any hermitian (or even normal) matrix  $\mathbf{T}$

$$\mu_1(\mathbf{T}) = \max_{f : \|f\|_2=1} \langle \mathbf{T}f, f \rangle. \quad (65)$$

We will use formula (65) to bound the first eigenvalue of operators  $\mathbf{T}_A^g$ .

Finally, it is easy to see that  $\mathbf{T}_A^B$  is a submatrix of the adjacency matrix of Caylay graph of  $A$  with minus and  $\tilde{\mathbf{T}}_A^B$  is an ordinary submatrix. Thus the operators  $\mathbf{T}_A^g$ ,  $\tilde{\mathbf{T}}_A^g$  can be considered as submatrices of the adjacency matrix of Caylay graph of  $A$  with some weights.

Let us consider several examples of operators.

Our **main example** the hermitian operator  $\mathbf{T}_A^{A \circ A}$  which we denote by  $\mathbf{T}$  for brevity. In the case, by variational principle (65) one has

$$\mu_1(\mathbf{T}) \geq \langle \mathbf{T}A(x)/|A|^{1/2}, A(x)/|A|^{1/2} \rangle = \mathbf{E}(A)|A|^{-1}. \quad (66)$$

Thus there is an obvious connection between the main eigenvalue of the operator and the additive energy. Moreover, using formula (64), we see that

$$\sum_{\alpha=1}^{|A|} \mu_\alpha^2(\mathbf{T}) = \sum_x (A \circ A)^3(x) = \mathbf{E}_3(A). \quad (67)$$

Hence the energies  $\mathbf{E}(A)$  and  $\mathbf{E}_3(A)$  being the second and the third moments of the convolution of the characteristic function are connected to each other much more closely than just by the Hölder inequality. Namely, they are connected through the operator  $\mathbf{T}$  and can be expressed via the eigenvalues of  $\mathbf{T}$ .

Another simple observation is the following. Consider a rectangular matrix  $M(\vec{x}, y) = A(y)A^k(\vec{x} + \Delta(y))$ ,  $\vec{x} \in A^k$ ,  $y \in A$ . It is easy to see that the *rectangular norm* of the matrix, that is

$$\|M\|_{\square}^4 := \sum_{\vec{x}, \vec{x}'} \sum_{y, y'} M(\vec{x}, y) \overline{M(\vec{x}', y)} \overline{M(\vec{x}, y')} M(\vec{x}', y')$$

equals  $E_{2k+1}(A)$ . On the other hand, it is well-known that the rectangular norm is the fourth moment of the singular values of  $M$  or, equivalently, the sum of squares of the eigenvalues of the square matrix  $(MM^*)(y, y') = (A \circ A)^k(y - y')A(y)A(y')$ . Taking  $k = 1$ , we obtain our operator  $T$ . Thus  $T$  is just the symmetrization of the matrix  $M(\vec{x}, y)$ , which is naturally and directly connected with higher energies. Another (dual) view on such operators can be found in [27].

Let us formulate the main technical proposition of the section.

**Proposition 34** *Let  $A \subseteq \mathbf{G}$  be a set,  $g_1, g_2$  be functions such that  $\overline{g_1(-x)} = g_1(x)$ , and  $\{f_\alpha\}$  be the orthonormal family of the eigenfunctions of the operator  $T_A^{g_1}$ . Then*

$$\sum_{x, y, z \in A} g_1(x - y) \overline{g_1(x - z)} \overline{g_2(y - z)} = \sum_{s, t} g_1(s) \overline{g_1(t)} \overline{g_2(t - s)} \mathcal{C}_3(A)(-s, -t) = \quad (68)$$

$$= \sum_{\alpha=1}^{|A|} \mu_\alpha^2(T_A^{g_1}) \cdot \overline{\langle T_A^{g_2} f_\alpha, f_\alpha \rangle}. \quad (69)$$

In particular, if  $g_1 = g_2 = g$  then

$$\sum_{x, y, z \in A} g(x - y) \overline{g(x - z)} \overline{g(y - z)} = \sum_{\alpha=1}^{|A|} |\mu_\alpha(T_A^g)|^2 \mu_\alpha(T_A^g).$$

**Proof.** Let  $\sigma$  be the first sum from (68). Then the first identity in the formula can be obtained by the changing of the variables  $s = x - y$ ,  $t = x - z$  and noting that the number of triples  $(x, y, z) \in A^3$  with  $s = x - y$ ,  $t = x - z$  is exactly  $\mathcal{C}_3(A)(-s, -t)$ .

Let us prove identity (69). It is easy to see that

$$\sigma = \sum_{x, y, z} T_A^{g_1}(x, y) \overline{T_A^{g_1}(x, z)} \overline{T_A^{g_2}(y, z)}. \quad (70)$$

By formula (62), we have

$$T_A^{g_1}(x, y) = \sum_{\alpha=1}^{|A|} \mu_\alpha f_\alpha(x) \overline{f_\alpha(y)}.$$

Substituting the last formula into (70) and changing the order of the summation, we obtain

$$\sigma = \sum_{y, z} \overline{T_A^{g_2}(y, z)} \sum_x \left( \sum_{\alpha=1}^{|A|} \mu_\alpha f_\alpha(x) \overline{f_\alpha(y)} \right) \left( \sum_{\alpha=1}^{|A|} \overline{\mu_\alpha f_\alpha(x)} f_\alpha(z) \right) =$$

$$= \sum_{y,z} \overline{T_A^{g_2}(y,z)} \sum_{\alpha,\beta} \mu_\alpha \overline{\mu_\beta} \overline{f_\alpha(y)} f_\beta(z) \sum_x f_\alpha(x) \overline{f_\beta(x)}.$$

By the orthonormality of the eigenfunctions, we get

$$\sigma = \sum_\alpha |\mu_\alpha|^2 \sum_{y,z} \overline{T_A^{g_2}(y,z)} \overline{f_\alpha(y)} f_\alpha(z) = \sum_\alpha |\mu_\alpha|^2 \langle \overline{T_A^{g_2} f_\alpha}, f_\alpha \rangle.$$

This completes the proof.  $\square$

**Exercise 35** *Using an appropriate generalization of the proposition above onto several operators, prove formula (42) of the previous section.*

Now we give two examples of operators with known spectrums and eigenfunctions.

**Example I.**

Let  $A \subseteq \mathbf{G}$  be a set,  $D = A - A$ ,  $S = A + A$ , and put weight  $g(x)$  equals  $D(x)$  or  $S(x)$ . In the following lemma we find, in particular, all eigenvalues as well as all eigenfunctions of operators  $T_A^D$ ,  $\tilde{T}_A^S$ .

**Lemma 36** *Let  $A \subseteq \mathbf{G}$  be a set,  $D = A - A$ ,  $S = A + A$ . Then the main eigenvalues and eigenfunctions of the operators  $T_A^D$ ,  $\tilde{T}_A^S$  equal  $\mu_1 = |A|$ , and  $f_1(x) = A(x)/|A|^{1/2}$ . All other eigenvalues equal zero and one can take for the correspondent eigenfunctions any orthonormal family of functions on  $A$  with zero mean.*

**Proof.** We have

$$A(x)(D * A)(x) = A(x)|D \cap (A - x)| = |A|A(x)$$

and thus by formula (59), we see that  $|A|$  is an eigenvalue of  $T_A^D$  and  $A(x)/|A|^{1/2}$  is the correspondent eigenfunction. Applying (64), we have

$$|A|^2 + \sum_{\alpha=2}^{|A|} |\mu_\alpha^2(T_A^D)| = \sum_{\alpha=1}^{|A|} |\mu_\alpha^2(T_A^D)| = \sum_x D(x)(A \circ A)(x) = |A|^2$$

and hence all other eigenvalues of  $T_A^D$  equal zero. The arguments for  $\tilde{T}_A^S$  are similar. This concludes the proof.  $\square$

Now we adapt the arguments from [29], see Proposition 28.

**Lemma 37** *Let  $A \subseteq \mathbf{G}$  be a finite set,  $D = A - A$ ,  $S = A + A$ . Suppose that  $\psi$  be a function on  $\mathbf{G}$ . Then*

$$|A|^2 \left( \sum_x \psi(x)(A \circ A)(x) \right)^2 \leq E_3(A) \sum_x |\psi(x)|^2 (D \circ D)(x), \quad (71)$$

and

$$|A|^2 \left( \sum_x \psi(x)(A \circ A)(x) \right)^2 \leq E_3(A) \sum_x |\psi(x)|^2 (S \circ S)(x). \quad (72)$$

*Proof.* Let us prove (72), the proof of (71) is similar. Applying Proposition 34 with  $g_2(x) = \psi(x)$  and  $g_1(x) = D(x)$ , we obtain in view of Lemma 36 that

$$\sum_{s,t} D(s)D(t)\psi(t-s)\mathcal{C}_3(A)(-s,-t) = |A|^{-1}\mu_1^2(\mathbf{T}_A^D)\langle \mathbf{T}_A^\psi A, A \rangle = |A| \sum_x \psi(x)(A \circ A)(x).$$

Using the Cauchy–Schwarz inequality and formula (36) of Corollary 21, we get

$$|A|^2 \left( \sum_x \psi(x)(A \circ A)(x) \right)^2 \leq \mathbf{E}_3(A) \sum_x |\psi(x)|^2 (D \circ D)(x)$$

as required.  $\square$

**Exercise 38** Obtain formulae (71), (72), using elementary arguments.

**Corollary 39** For any  $A \subseteq \mathbf{G}$  the following holds

$$|A|^2 \mathbf{E}_{3/2}^2(A) \leq \mathbf{E}_3(A) \mathbf{E}(A, A \pm A). \quad (73)$$

This inequality was obtained in [15] and implies formula (56) from Lecture 2 without any additional multiplicative constants.

**Corollary 40** For any  $A \subseteq \mathbf{G}$  the following holds

$$|A|^6 \leq \mathbf{E}_3(A) \cdot \sum_{x \in A-A} ((A \pm A) \circ (A \pm A))(x).$$

This is an inequality from [24].

### Example II.

Let  $p$  be a prime number,  $q = p^s$  for some integer  $s \geq 1$ . Let  $\mathbb{F}_q$  be the field with  $q$  elements, and let  $\Gamma \subseteq \mathbb{F}_q^*$  be a multiplicative subgroup. We will write  $\mathbb{F}_q^*$  for  $\mathbb{F}_q \setminus \{0\}$ . Denote by  $t$  the cardinality of  $\Gamma$ , and put  $n = (q-1)/t$ . Let also  $\mathbf{g}$  be a primitive root, then  $\Gamma = \{\mathbf{g}^{nl}\}_{l=0,1,\dots,t-1}$ . Let  $\{\chi_\alpha(x)\}_{\alpha \in [t]}$  be the orthogonal family of multiplicative characters on  $\Gamma$  and  $\{f_\alpha(x)\}_{\alpha \in [t]}$  be the correspondent orthonormal family, that is

$$f_\alpha(x) = |\Gamma|^{-1/2} \chi_\alpha(x) = |\Gamma|^{-1/2} \cdot e\left(\frac{\alpha l}{t}\right), \quad x = \mathbf{g}^{nl}, \quad 0 \leq l < t. \quad (74)$$

In particular,  $f_\alpha(x) = \chi_\alpha(x) = 0$  if  $x \notin \Gamma$ . Clearly, products of such functions form a basis on Cartesian products of  $\Gamma$ .

If  $\varphi : \Gamma \rightarrow \mathbb{C}$  be a function then denote by  $c_\alpha(\varphi)$  the correspondent coefficients of  $\varphi$  relatively to the family  $\{f_\alpha(x)\}_{\alpha \in [t]}$ . In other words,

$$c_\alpha(\varphi) := \langle \varphi, f_\alpha \rangle = \sum_{x \in \Gamma} \varphi(x) \overline{f_\alpha(x)}, \quad \alpha \in [|\Gamma|].$$

In the next lemma we calculate, in particular, the spectrums of all operators with  $\Gamma$ -invariant weights  $g$  (that is  $g(x\gamma) = g(x)$  for all  $\gamma \in \Gamma$ .) The lemma was proved mainly in [26]. We give the proof for the sake of completeness. Further results on the spectrum of operators connected with multiplicative subgroups can be found in [31].

**Lemma 41** *Let  $\Gamma \subseteq \mathbb{F}_q^*$  be a multiplicative subgroup. Suppose that  $H(x, y) : \Gamma \times \Gamma \rightarrow \mathbb{C}$  satisfies two conditions*

$$H(y, x) = \overline{H(x, y)} \quad \text{and} \quad H(\gamma x, \gamma y) = H(x, y), \quad \forall \gamma \in \Gamma. \quad (75)$$

*Then the functions  $\{f_\alpha(x)\}_{\alpha \in [\Gamma]}$  form the complete orthonormal family of the eigenfunctions of the operator  $H(x, y)$ .*

**Proof.** The first property of (75) says that  $H$  is a hermitian operator, so it has a complete orthonormal family of its eigenfunctions. Consider the equation

$$\mu f(x) = \Gamma(x) \sum_{y \in \Gamma} H(x, y) f(y), \quad (76)$$

where  $\mu$  is some number and  $f : \Gamma \rightarrow \mathbb{C}$  is unknown function. It is sufficient to check that any  $f = \chi_\alpha$ ,  $\alpha \in [\Gamma]$  satisfies the equation above. Indeed, making a substitution  $x \rightarrow x\gamma$  into (76) and using the characters property, we obtain

$$\mu f(x) f(\gamma) = \Gamma(x\gamma) \sum_y H(\gamma x, y) f(y) = \Gamma(x) \sum_y H(\gamma x, \gamma y) f(\gamma y) = \Gamma(x) f(\gamma) \sum_y H(x, y) f(y),$$

where the second property of (75) has been used. Thus, it remains to check (76) just for one  $x \in \Gamma$ . Choosing the number  $\mu$  in an appropriate way we attain the former. This completes the proof.  $\square$

**Corollary 42** *Let  $\Gamma$  be a multiplicative subgroup. Then  $\mu_1(\Gamma) = E(\Gamma)|\Gamma|^{-1}$  and*

$$E(\Gamma) = \max_{f : \|f\|_2=1, \text{supp } f \subseteq \Gamma} E(\Gamma, f).$$

**Proof.** Indeed by Lemma 41 the function  $f(x) := \Gamma(x)/|\Gamma|^{1/2}$  is the main eigenfunction of any operator  $T_\Gamma^g$  with  $\Gamma$ -invariant function  $g(x)$ . In particular,  $\langle Tf, f \rangle = E(\Gamma)|\Gamma|^{-1}$ . The second formula follows from variational principle (65).  $\square$

**Exercise 43** *Let  $\Gamma$  be a multiplicative subgroup. Then for any  $A \subseteq \Gamma$  one has*

$$E(\Gamma) \cdot \frac{|A|^2}{|\Gamma|^2} \leq E(A, \Gamma),$$

and

$$E^2(A, \Gamma)|\Gamma| \leq E_3(\Gamma)E^\times(A).$$

We give a number-theoretical application of Lemma 41 above. *Heilbronn's exponential sum* is defined by

$$S(a) = \sum_{n=1}^p e^{2\pi i \cdot \frac{an^p}{p^2}}. \quad (77)$$



D.R. Heath–Brown obtained in [6] the first nontrivial upper bound for the sum,  $a \neq 0$ . After that the result was improved in papers [7], [28], [34], [32].

Let us prove the best upper bound for the sum  $S(a)$ , see [32].

Consider the following multiplicative subgroup

$$\Gamma = \{m^p : 1 \leq m \leq p-1\} = \{m^p : m \in \mathbb{Z}/(p^2\mathbb{Z}), m \neq 0\} \subseteq \mathbb{Z}/(p^2\mathbb{Z}) \quad (78)$$

and note that  $S(a)$  is just a sum over the subgroup  $\Gamma$ . We need in a lemma, which is analog of Corollary 32.

**Lemma 44** *For Heilbronn's subgroup (78), one has*

$$E_3(\Gamma) \ll p^3 \log p. \quad (79)$$

*Proof.* Arranging  $(\Gamma \circ \Gamma)(x_1) \geq (\Gamma \circ \Gamma)(x_2) \geq \dots$ , where  $x_j$  belong to different cosets, we have by Lemma 7 from [7] (see also Lemma 5 from [28]) that

$$(\Gamma \circ \Gamma)(x_j) \ll |\Gamma|^{2/3} j^{-1/3}.$$

Thus

$$E_3(\Gamma) = \sum_x (\Gamma \circ \Gamma)^3(x) \ll |\Gamma|^3 + |\Gamma| \sum_j (|\Gamma|^{2/3} j^{-1/3})^3 \ll p^3 \log p$$

as required.  $\square$

**Theorem 45** *Let  $p$  be a prime, and  $a \neq 0 \pmod{p}$ . Then*

$$|S(a)| \ll p^{\frac{5}{6}} \log^{\frac{1}{6}} p. \quad (80)$$

*Proof.* Put  $t = |\Gamma| = p-1$ . There is  $\xi \neq 0$  such that

$$M^2 := |S(a)|^2 = t^{-1} \sum_{x \in \xi\Gamma} |\widehat{\Gamma}(x)|^2 = \langle T_{\Gamma}^{\widehat{\xi\Gamma}} \Gamma(x)/t^{-1/2}, \Gamma(x)/t^{-1/2} \rangle. \quad (81)$$

To derive the last identity we have used formulae (6), (60). Consider the operator  $T_{\Gamma}^{\widehat{\xi\Gamma}}$ . Applying the Fourier transform or just simple calculations, it is easy to see that the operator is nonnegatively defined. In view of Lemma 41 and identity (81), we obtain  $\mu_1(T_{\Gamma}^{\widehat{\xi\Gamma}}) = M^2$ . Using Proposition 34 with  $g_1 = g_2 = \xi\Gamma$ , further, nonnegativity of the operator  $T_{\Gamma}^{\widehat{\xi\Gamma}}$ , Corollary 21 and the Cauchy–Schwarz inequality, we get

$$M^2 \leq E_3(\Gamma) \sum_{a,b} |\widehat{\xi\Gamma}(a)|^2 |\widehat{\xi\Gamma}(b)|^2 |\widehat{\xi\Gamma}(a-b)|^2.$$

Finally, applying formula (44) and Lemma 44, we obtain

$$M^{12} \leq E_3^2(\Gamma)p^2 \ll t^6 p^2 \log^2 t,$$

and inequality (80) follows.  $\square$

In two examples above we know eigenfunctions of the considered operators. In our main example of the operator  $T$  such functions are usually unknown. Nevertheless, one can obtain some results in this general situation.

We start with a lemma from [30], which shows that the operator  $T_A^{A \circ A}$  somehow "feels" another operators  $T_A^g, \tilde{T}_A^g$  for "regular" weights  $g$ .

**Lemma 46** *Let  $A \subseteq \mathbf{G}$  be a set and  $g$  be a nonnegative function on  $\mathbf{G}$ . Suppose that  $f_1$  is the main eigenfunction of  $T_A^g$  or  $\tilde{T}_A^g$ , and  $\mu_1$  is the correspondent eigenvalue. Then*

$$\langle T_A^{A \circ A} f_1 f_1 \rangle \geq \frac{\mu_1^3}{\|g\|_2^2 \cdot \|g\|_\infty}.$$

*Proof.* By assumption  $g$  is a nonnegative function on  $\mathbf{G}$ . It implies that  $f_1$  is also a nonnegative function. We have

$$\mu_1 f_1(x) = A(x)(g * f_1)(x). \quad (82)$$

Thus

$$\mu_1^2 \left( \sum_x f_1(x) \right)^2 \leq \left( \sum_x g(x)(f_1 \circ A)(x) \right)^2 \leq \|g\|_2^2 E(A, f_1) = \|g\|_2^2 \langle T_A^{A \circ A} f_1 f_1 \rangle. \quad (83)$$

On the other hand, returning to (82) and using  $\|f\|_2 = 1$ , we get

$$\mu_1 = \sum_x (f_1 \circ f_1)(x) \leq \|g\|_\infty \left( \sum_x f_1(x) \right)^2.$$

Substituting the last estimate into (83), we obtain the result.  $\square$

By formula (66) we know that  $\mu_1(T) \geq E(A)/|A|$ . The next lemma shows that a similar upper bound holds if one consider large subsets of  $A$ .

**Lemma 47** *Let  $A \subseteq \mathbf{G}$  be a set. There is  $A' \subseteq A$ ,  $|A'| \geq |A|/2$  such that  $\mu_1(T_{A'}^P) \leq \frac{2E(A)}{\Delta|A|}$  for any set  $P \subseteq \{x : |A_x| \leq \Delta\}$  and any real number  $\Delta > 0$ . In particular,  $\mu_1(T_{A'}^{A \circ A}) \leq \frac{2E(A)}{|A|}$ .*

*Proof.* Let

$$A_1 = \{x : ((A * A) \circ A)(x) > 2E(A)/|A|\}.$$

It is easy to see that  $|A_1| < |A|/2$ . Put  $A' = A \setminus A_1$  and let  $f$  be the main eigenfunction of the operator  $T_{A'}^P$ . Let also  $\mu_1 = \mu_1(T_{A'}^P)$ . We have

$$\mu_1 f(x) = A'(x)(P * f)(x).$$

Summing over  $x \in A'$  and using the definition of the set  $A'$ , we obtain

$$\begin{aligned} \mu_1 \sum_x f(x) &= \sum_x f(x)(P \circ A')(x) \leq \Delta^{-1} \sum_x f(x)((A \circ A) \circ A)(x) = \\ &= \Delta^{-1} \sum_x f(x)((A * A) \circ A)(x) \leq \Delta^{-1} \frac{2E(A)}{|A|} \cdot \sum_x f(x) \end{aligned}$$

and we are done.  $\square$

There is an important class of so-called connected sets. Formally, let  $\beta, \gamma \in [0, 1]$ . A set  $A \subseteq \mathbf{G}$  is called  $(\beta, \gamma)$ -connected if for any  $B \subseteq A$ ,  $|B| \geq \beta|A|$  the following holds

$$E(B) \geq \gamma \left( \frac{|B|}{|A|} \right)^4 E(A).$$

Using Lemma 47, one can obtain an unusual relation between energies  $E_s(A)$ ,  $s \in [1, 2]$  and  $E(A)$  for any connected set  $A$ , see [30].

**Exercise 48** Let  $A \subseteq \mathbf{G}$  be a set, and  $\beta, \gamma \in [0, 1]$ . Suppose that  $A$  is  $(\beta, \gamma)$ -connected with  $\beta \leq 1/2$ . Further for any  $s \in [1, 2]$  the following holds

$$E_s(A) \geq 2^{-5} \gamma |A|^{1-s/2} E^{s/2}(A). \quad (84)$$

Using lemmas above we can formulate the second main result of the section (another theorems of such type can be found in [2], [30]). By formulas (66), (67) we know that the energies  $E(A)$ ,  $E_3(A)$  are connected through the operator  $T$ . It allows us give a full description of sets  $A$  having "critical relation" between  $E(A)$ ,  $E_3(A)$  that is  $E_3(A) \ll E^2(A)/|A|^2$ . Namely, inequality  $E_3(A) \ll E^2(A)/|A|^2$  holds iff  $A$  contains a large subset  $A'$  such that  $|nA' - mA'| \approx |A' - A'|$  for any positive integers  $n, m$ . Informally, it says that the growth of the size of the sumset  $kA'$  of the set  $A'$  stops after the second step. More precisely, the following holds (previous results in the direction can be found in [26] and [29]).

**Theorem 49** Let  $A \subseteq \mathbf{G}$  be a set,  $E(A) = |A|^3/K$ , and  $E_3(A) = M|A|^4/K^2$ . Then there is a set  $A' \subseteq A$  such that

$$|A'| \gg M^{-10} \log^{-15} M \cdot |A|, \quad (85)$$

and

$$|nA' - mA'| \ll (M^9 \log^{14} M)^{6(n+m)} K |A'| \quad (86)$$

for every  $n, m \in \mathbb{N}$ .

**Proof.** Let  $E = E(A) = |A|^2/K$ ,  $E_3 = E_3(A)$ ,  $L = 2 \log(4M)$ . Write

$$D_j = \{x \in A - A : 2^{j-2}|A|K^{-1} < |A_x| \leq 2^{j-1}|A|K^{-1}\}.$$

Trivially

$$|D_j|(2^{j-2}|A|K^{-1})^3 \leq E_3,$$

and whence

$$|D_j| \ll \mathbb{E}_3 / (|A|^3 K^{-3} 2^{3j}). \quad (87)$$

Thus

$$\mathbb{E} \ll \sum_{j=1}^l \sum_s |A_s|^2,$$

where  $l$  can be estimated as  $\log M \leq L$ . By pigeonhole principle we find  $j \in [l]$  such that

$$L^{-1} \mathbb{E} \ll \sum_{s \in D_j} |A_s|^2. \quad (88)$$

Put  $D = D_j$ ,  $\Delta = 2^{j-1} |A| K^{-1}$ , and  $g(x) = (A \circ A)(x) D(x)$ . From (88) it follows that

$$|D| \gg \frac{|A| K}{L M^2} \quad (89)$$

and

$$\sum_{x \in D} (A \circ A)(x) \gg \frac{|A|^2}{L M}. \quad (90)$$

Consider the operators  $T_1 = T_A^g$ ,  $T_2 = T_{A,D}^A$  and  $T_3 = T_A^{A \circ A}$ . Using Lemma 46, we get

$$\langle T_3 f_1, f_1 \rangle \geq \frac{\mu_0^3(T_1)}{\|g\|_2^2 \|g\|_\infty} \gg \frac{|D|^2 \Delta^3}{|A|^3} := \sigma. \quad (91)$$

Clearly, all elements of matrices  $T_1, (T_2)^* T_2$  does not exceed elements of  $T_3$  and the operator  $T_3$  is nonnegative defined. By formula (88), we have

$$\frac{\mathbb{E}}{4L|A|} \leq \mu_1(T_1). \quad (92)$$

Similarly,

$$\frac{\mathbb{E}}{4L|A|} \leq \mu_1(T_1) \leq \langle T_3 f_1, f_1 \rangle, \quad (93)$$

where  $f_1 \geq 0$  is the main eigenfunction of the operator  $T_1$ . Applying Proposition 34 with  $A = A$ ,  $g_1 = g$ ,  $g_2 = A \circ A$ , we obtain

$$\begin{aligned} \mu_1^4(T_1) \sigma^2 &\ll \mathbb{E}_3 \sum_{\alpha \in D, \beta \in D : (A \circ A)(\alpha - \beta) \geq d} (A \circ A)^2(\alpha) (A \circ A)^2(\beta) (A \circ A)^2(\alpha - \beta) \\ &\ll \mathbb{E}_3 \Delta^4 \sum_{x : (A \circ A)(x) \geq d} (D \circ D)(x) (A \circ A)^2(x), \end{aligned} \quad (94)$$

(where  $d$  can be taken as  $d = \frac{\mu_0^2(T_1)}{32|A|\mathbb{E}_3^{1/2}}$ ). Applying the Cauchy–Schwartz inequality, we have

$$\sum_x (A \circ A)^2(x) (D \circ D)(x) \leq \mathbb{E}_3^{2/3} \left( \sum_x (D \circ D)^3(x) \right)^{1/3} \leq \mathbb{E}_3^{2/3} |D|^{1/3} \mathbb{E}^{1/3}(D).$$

Put  $E(D) = \mu|D|^3$ . Recalling (94), we get

$$\mu_1^4(T_1)\sigma^2 \ll \left(\frac{M|A|^4}{K^2}\right)^{5/3} \Delta^4 |D|^{4/3} \mu^{1/3}. \quad (95)$$

We have  $\Delta \ll M|A|/K$ . In the situation the following holds  $\sigma \geq \mu_1(T_1)$ . Thus, an accurate calculations give

$$E(D) = \mu|D|^3 \gg \frac{|D|^3}{M^9 L^{14}}.$$

By Balog–Szemerédi–Gowers Theorem, see e.g. [38], there is  $D' \subseteq D$  such that  $|D'| \gg \mu|D|$  and  $|D' + D'| \ll \mu^{-6}|D'|$ . Plünnecke–Ruzsa inequality (see [16] or again [38]) yields

$$|nD' - mD'| \ll \mu^{-6(n+m)}|D'|, \quad (96)$$

for every  $n, m \in \mathbb{N}$ . Using the definition of the set  $D = D_j$  and inequality (90), we find  $x \in \mathbf{G}$  such that

$$|(A - x) \cap D'| \gg \mu|A|L^{-1}M^{-1} \gg M^{-10}L^{-15} \cdot |A|. \quad (97)$$

Put  $A' = A \cap (D' + x)$ . Using (96), (97) and the definition of  $\Delta$ , we obtain for all  $n, m \in \mathbb{N}$

$$|nA' - mA'| \leq |nD' - mD'| \ll \mu^{-7(n+m)}|A||A'|\Delta^{-1} \ll \mu^{-6(n+m)}K|A'| \quad (98)$$

and the theorem is proved.  $\square$

**Remark 50** For every convex set Theorem 49 above easily gives a "nontrivial" estimate  $E(A) \ll |A|^{5/2-\varepsilon_0}$ , where  $\varepsilon_0 > 0$  is an absolute constant. Indeed, suppose that  $E(A) \gg |A|^{5/2-\varepsilon}$  and  $\varepsilon > 0$  is sufficiently small. Then recalling the bound  $E_3(A) \ll |A|^3 \log |A|$ , we see that in terms of Theorem 49 one has  $M = \log |A|$ . So,  $M$  is small and we can effectively apply the theorem. Thus there is a set  $A' \subseteq A$  from Theorem 49 such that

$$|A|^{7/4} \ll_M |A' + A' - A'| \ll_M |A|^4 E^{-1}(A) \quad (99)$$

and the result follows. In the derivation of the first inequality of (99) we have used formula (53) of Exercise 30.

Applying more refine method from [24] one can get even simpler proof. Indeed, for so large  $A' \subseteq A$  we have (see Exercise 30) that  $|A|^{3/2+\varepsilon_1} \ll_M |A' - A'| \ll_M |A|^4 E^{-1}(A)$ , where  $\varepsilon_1 > 0$  is an absolute constant. Again we obtain a lower bound for  $\varepsilon_0$ . Interestingly, that lower bounds for the doubling constants give us upper bounds for the additive energy in the case. Of course our real arguments even more direct and they give a concrete bound

$$E(A) \ll |A|^{32/13+\varepsilon}, \quad \varepsilon > 0 \quad (100)$$

for any convex set  $A$ .

The same proof takes place for multiplicative subgroups  $\Gamma \subseteq \mathbb{Z}/p\mathbb{Z}$ , where  $p$  is a prime number if one use Stepanov's method (see e.g. [13] or [35]) or combine Stepanov's method with recent lower bounds for the doubling constant of subgroups from [26, 35, 29]. The direct arguments give (100) for any subgroup  $\Gamma$  of size less than  $\sqrt{p}$ , say.

Theorem 49 describes all sets  $A$ , having "critical" relation between energies  $E_2(A)$ ,  $E_3(A)$ . Another pairs of energies were considered in papers [2] and [30].

## References

- [1] N. ALON, B. BUKH, B. SUDAKOV, *Discrete Kakeya-type problems and small bases*, Israel J. Math. 174 (2009), 285–301.
- [2] M. Bateman, N. Katz, *Structure in additively nonsmoothing sets*, arXiv:1104.2862v1 [math.CO] 14 Apr 2011.
- [3] E. CROOT, O. SISASK, *A probabilistic technique for finding almost-periods of convolutions*, Geom. Funct. Anal. 20 (2010), 1367–1396.
- [4] G. ELEKES, M. NATHANSON, I. Z. RUZSA, *Convexity and sumsets*, J. Number Theory 83 (2000), 194–201.
- [5] M. Z. GARAEV, *On lower bounds for  $L_1$ -norm of exponential sums*, Mathematical Notes 68 (2000), 713–720.
- [6] D. R. HEATH-BROWN, *An estimate for Heilbronn’s exponential sum*, Analytic number theory vol. 2, (Allerton Park, IL 1995), Progr. Math., 1 39, Birkhäuser, Boston (1996), 451–463.
- [7] D. R. HEATH-BROWN, S. V. KONYAGIN, *New bounds for Gauss sums derived from  $k$ th powers, and for Heilbronn’s exponential sum*, Quart. J. Math. 51 (2000), 221–235.
- [8] N. HEGYVÁRI, *On consecutive sums in sequences*, Acta Math. Acad. Sci. Hungar. 48 (1986), 193–200.
- [9] R. HORN, C. JOHNSON, *Matrix Analysis*, Cambridge University Press, Cambridge, 1985, xiii+561 pp.
- [10] J. JOHNSEN, *On the distribution of powers in finite fields*, J. Reine Angew. Math., 251, 1971, 10–19.
- [11] N. H. KATZ, P. KOESTER, *On additive doubling and energy*, SIAM J. Discrete Math., 24(4) : 1684–1693, 2010.
- [12] V. S. KONYAGIN, *An estimate of  $L_1$ -norm of an exponential sum*, The Theory of Approximations of Functions and Operators. Abstracts of Papers of the International Conference Dedicated to Stechkin’s 80th Anniversay [in Russian]. Ekaterinburg (2000), 88–89.
- [13] S. V. KONYAGIN, *Estimates for trigonometric sums and for Gaussian sums*, IV International conference ”Modern problems of number theory and its applications”. Part 3 (2002), 86–114.
- [14] S. KOPPARTY, V .F. LEV, S. SARAF, M. SUDAN, *Kakeya-type sets in finite vector spaces*, arXiv:1003.3736v1 [math.NT].
- [15] L. LI, *On a theorem of Schoen and Shkredov on sumsets of convex sets*, arXiv:1108.4382v1 [math.CO].

- [16] G. PETRIDIS, *New Proofs of Plünnecke-type Estimates for Product Sets in Non-Abelian Groups*, *Combinatorica*, 1–14, 2012.
- [17] O.E. RAZ, O. ROCHE–NEWTON, M. SHARIR, *Sets with few distinct distances do not have heavy lines*, *arXiv:1410.1654v1 [math.CO]* 7 Oct 2014.
- [18] W. RUDIN, *Fourier analysis on groups*, Wiley 1990 (reprint of the 1962 original).
- [19] I.Z. RUZSA, *Sumsets and structure*, *Combinatorial number theory and additive group theory* (2009): 87–210.
- [20] T. SANDERS, *The structure theory of sets addition revisited*, *Bull. Amer. Math. Soc. (N.S.)*, **50**(1):93–127, 2013.
- [21] T. SANDERS, *On the Bogolyubov-Ruzsa lemma*, *Anal. PDE*, **5**(3):627–655, 2012.
- [22] T. SANDERS, *On Roth’s theorem on progressions*, *Ann. of Math. (2)*, **174**(1):619–636, 2011.
- [23] T. SCHOEN, *Near optimal bounds in Freiman’s theorem*, *Duke Math. J.*, **158**:1–12, 2011.
- [24] T. SCHOEN AND I. SHKREDOV, *Additive properties of multiplicative subgroups of  $\mathbb{F}_p$* , *Q. J. Math.* **63** (2012), no. 3, 713–722.
- [25] T. SCHOEN AND I. SHKREDOV, *On sumsets of convex sets*, *Combin. Probab. Comput.* **20** (2011), no. 5, 793–798.
- [26] T. SCHOEN AND I. SHKREDOV, *Higher moments of convolutions*, *J. Number Theory* **133** (2013), no. 5, 1693–1737.
- [27] I. D. SHKREDOV, *Some applications of W. Rudin’s inequality to problems of combinatorial number theory*, *Uniform Distribution Theory*, **6**:2 (2011), 95–116.
- [28] I. D. SHKREDOV, *On Heilbronn’s exponential sum*, *Quart. J. Math.*, (2012), 1–10, doi: 10.1093/qmath/has037.
- [29] I. D. SHKREDOV, *Some new inequalities in additive combinatorics*, *MJCNT*, **3**:2 (2013), 237–288.
- [30] I. D. SHKREDOV, *Some new results on higher energies*, *Transactions of MMS*, **74**:1 (2013), 35–73.
- [31] I.D. SHKREDOV, *Energies and structure of additive sets*, *Electronic Journal of Combinatorics*, **21**(3) (2014), #P3.44, 1–53.
- [32] I. D. SHKREDOV, *On exponential sums over multiplicative subgroups of medium size*, *Finite Fields and Their Applications* **30** (2014), 72–87.
- [33] I. D. SHKREDOV, *On sums of Szemerédi–Trotter sets*, *Transactions of Steklov Mathematical Institute*, **289**, (2015), 300–309.

- [34] I. D. SHKREDOV, E. V. SOLODKOVA, I. V. VYUGIN, *Intersections of multiplicative subgroups and Heilbronns exponential sum*, arXiv:1302.3839v3 [math.NT] 6 May 2015.
- [35] I. D. SHKREDOV AND I. V. VYUGIN, *On additive shifts of multiplicative subgroups*, Sb. Math. 203 (2012), no. 5-6, 844–863.
- [36] J. SOLYMOSI, *Sumas contra productos*, Gaceta de la Real Sociedad Matematica Espanola, ISSN 1138–8927, 12 (2009).
- [37] E. SZEMERÉDI, W. T. TROTTER, *Extremal problems in discrete geometry*, Combinatorica 3 (1983), 381–392.
- [38] T. TAO, V. VU, *Additive combinatorics*, Cambridge University Press 2006.

I.D. Shkredov  
Steklov Mathematical Institute,  
ul. Gubkina, 8, Moscow, Russia, 119991  
and  
IITP RAS,  
Bolshoy Karetny per. 19, Moscow, Russia, 127994  
ilya.shkredov@gmail.com